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Comparison of Two Reformulation-Linearization Technique Based Linear Programming Relaxations for Polynomial Programming Problems

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Abstract. In this paper, we compare two strategies for constructing linear programming relaxations for polynomial programming problems using a Reformulation-Linearization Technique (RLT). RLT involves an automatic reformulation of the problem via the addition of certain nonlinear implied constraints that are generated by using the products of the simple bounding restrictions (among other products), and a subsequent linearization based on variable redefinitions. We prove that applying RLT directly to the original polynomial program produces a bound that dominates in the sense of being at least as tight as the value obtained when RLT is applied to the joint collection of all equivalent quadratic problems that could be constructed by recursively defining additional variables as suggested by Shor.

Key words: Polynomial programming, reformulation-linearization technique, outer-approximations, tight linear programming relaxations.

1. Introduction

In this paper, we theoretically compare two lower bounding linear programming relaxations for polynomial programming problems, **PP**, where,

$$\textbf{PP}: \text{ Minimize}\{\phi_0(x) : x \in Z \cap \Omega\},$$

and where $Z = \{x : \phi_r(x) \geqslant \beta_r$ for $r = 1, \ldots, R_1$, $\phi_r(x) = \beta_r$ for $r = R_1 + 1, \ldots, R\}$, $\Omega = \{x : 0 \leqslant \ell_j \leqslant x_j \leqslant u_j < \infty$, for $j = 1, \ldots, n\}$ and $\phi_r(x) \equiv \sum_{t \in T_r} \alpha_{rt}[\pi_{j \in J_{rt}} x_j]$ for $r = 0, 1, \ldots, R$. Here, T_r is an index set for the terms defining $\phi_r(\cdot)$, and α_{rt} are real coefficients for the polynomial terms $(\pi_{j \in J_{rj}} x_j)$, $t \in T_r, r = 0, 1, \ldots, R$. Note that we permit a repetition of indices within each set J_{rt}. For example, if $J_{rt} = \{1, 2, 2, 3\}$, then the corresponding polynomial term is $x_1 x_2^2 x_3$. In particular, let us denote $N = \{1, \ldots, n\}$, and let δ be the maximum degree of any polynomial term appearing in PP. Define $\bar{N} = \{N, \ldots, N\}$ to be composed of δ replicates of N. Then, each $J_{rt} \subseteq \bar{N}$, with $1 \leqslant |J_{rt}| \leqslant \delta$, for $t \in T_r, r = 0, 1, \ldots, R$.

Problem PP belongs to the general class of constrained global optimization problems, for which Horst and Tuy [7] prescribe a variety of promising methods.

However, we focus here on a successive quadratic variable substitution strategy due to Shor [12] which is used to transform a given polynomial problem to an equivalent quadratically constrained quadratic problem. By using graph theoretic approaches, Hansen and Jaumard [4] improve Shor's quadratic transformation by reducing the number of additional variables. Floudas and Visweswaran [2,3], Al-Khayyal *et al.* [1], and Quesada and Grossman [9] adopt similar successive quadratic transformations to obtain an equivalent bilinearly constrained bilinear programming problem. For this problem, various lower bounding procedures based on generalized Benders' techniques, or convex/concave envelopes of the bilinear terms, along with other bound tightening strategies are proposed.

While this transformation produces a convenient quadratic representation, it is of interest to explore what the potential loss might be in this process. We examine this issue in light of the *Reformulation-Linearization Technique* (**RLT**) of Sherali and Tuncbilek [11] by studying the effect on the lower bounds produced when applying this technique directly to Problem PP, or alternatively to its equivalent quadrified form. The outcome of this comparison is not so obvious because the quadrification process generates additional classes of constraints to which RLT is applied. Moreover, this quadrification can result in many alternative, equivalent representations, and it is uncertain what might result if one were to simultaneously incorporate all possible representations in the quadratic transformation.

More specifically, the process of applying RLT to PP begins by generating implied constraints using *distinct* products of *bound factors* $(x_j - \ell_j) \geqslant 0$, $(u_j - x_j) \geqslant 0, j \in N$, taken δ at a time. After including these constraints in PP, the resulting problem is linearized by defining new variables, one for each distinct polynomial term, to obtain the following linear programming problem.

$$\mathbf{LP}(\mathbf{PP}): \quad \text{Minimize}[\phi_o(x)]_\ell \tag{1a}$$

$$\text{subject to } [\phi_r(x)]_\ell \geqslant \beta_r \quad \forall r = 1, \ldots, R_1,$$

$$[\phi_r(x)]_\ell = \beta_r$$

$$\forall r = R_1 + 1, \ldots, R \tag{1b}$$

$$\left[\underset{j \in J_1}{\pi}(x_j - \ell_j)\underset{j \in J_2}{\pi}(u_j - x_j)\right]_\ell \geqslant 0$$

$$\forall (J_1 \cup J_2) \subseteq \bar{N}, \quad |J_1 \cup J_2| = \delta \tag{1c}$$

where $[(\cdot)]_\ell$ denotes the linearized form of the polynomial function $(\cdot)$ that is obtained upon substituting a single variable for each distinct polynomial term according to

$$X_j = \underset{j \in J}{\pi} x_j \quad \forall J \subseteq \bar{N}. \tag{2}$$

Here, the indices in J are assumed to be sequenced in nondecreasing order, and each distinct set J produces one distinct X_J variable. Also, $X_{\{j\}} \equiv x_j \forall j \in N$, and $X_\emptyset \equiv 1$. Furthermore, note that throughout, by $J_1 \cup J_2$ we will mean the joint collection of indices appearing in J_1 and J_2, including repetitions. We also remark here that additional valid product constraints can be introduced in LP(PP) in order to tighten the relaxation. However, from the viewpoint of obtaining theoretical convergence, the product constraints generated in (1c) are sufficient.

Denoting the optimal objective function value of any problem $[\cdot]$ by $\nu[\cdot]$, Sherali and Tuncbilek [11] show that $\nu[LP(PP)] \leqslant \nu[PP]$. An alternative linear programming relaxation can be obtained by first applying Shor's quadrification method to PP, and subsequently, using RLT on the resulting equivalent quadratically constrained quadratic programming problem. In this paper, we prove that by applying RLT directly to the original polynomial program, we derive a bound that *dominates* (in the sense of being at least as tight as) the value obtained when RLT is applied to an all encompassing quadratic problem that is constructed by *simultaneously* or *jointly* using all possible quadrified representations of the original problem. Some related numerical comparisons are also presented to show that this dominance can be strict, and sometimes quite significant.

2. RLT Applied to Quadrified Polynomial Programs

First, let us present the basic transformation used for *quadrifying* Problem PP, i.e., for converting it into an equivalent quadratic polynomial program. Let the highest degree of each variable x_j appearing in PP be S_j, $j = 1, \ldots, n$, and define $s_j = \lceil S_j/2 \rceil$, where $\lceil \cdot \rceil$ denotes the rounding-up operation. Consider the set

$$A = \Big\{ a = (a_1, \ldots, a_n) \in Z_+^n : 0 \leqslant a_j \leqslant s_j \quad \forall j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} a_j \leqslant \delta \Big\} \tag{3}$$

where Z_n^+ is the set of nonnegative integral n-vectors, and for each $a \in A$, define a *variable* $R[a]$ to represent the *monomial* $x^a \equiv \pi_{j=1}^n x_j^{a_j}$. Our equivalent quadrified program will be defined in terms of the variables $R[a], a \in A$.

To illustrate the quadrification process, consider the polynomial term $x_1^3 x_2^2 x_3^4$. This can be reduced to a quadratic form by including the following series of identity relations involving the $R[\cdot]$ variables,

$$\begin{aligned}
x_1 &\equiv R[1,0,0], \quad x_2 \equiv R[0,1,0], \quad x_3 \equiv R[0,0,1],\\
x_3^2 &\equiv R[0,0,2] = R[0,0,1]R[0,0,1]\\
x_2 x_3^2 &\equiv R[0,1,2] = R[0,0,2]R[0,1,0],\\
x_1 x_2 x_3^2 &\equiv R[1,1,2] = R[0,1,2]R[1,0,0]\\
x_1^2 x_2 x_3^2 &\equiv R[2,1,2] = R[1,1,2]R[1,0,0],
\end{aligned}$$

and then, replacing $x_1^3 x_2^2 x_3^4$ by $R[1,1,2]R[2,1,2]$ in the $R[\cdot]$ variables.

Hence, by defining suitable variables $R[a], a \in A$, and by including appropriate identities involving these variables as done above, we can thereby quadrify Problem PP. Notice that this successive quadrification scheme is not uniquely defined; for example, instead of writing $x_2x_3^2$ as $(x_2)(x_3^2)$ as essentially done above, we could have as well composed $x_2x_3^2 = (x_3)(x_2x_3)$ as follows:

$$x_2x_3 \equiv R[0,1,1] = R[0,1,0]R[0,0,1],$$
$$x_2x_3^2 \equiv R[0,1,2] = R[0,1,1]R[0,0,1].$$

In order to simultaneously capture all such transformations or quadrifying identities within an *all-encompassing* equivalent quadratic polynomial program, and then to apply RLT to this quadratic program, we adopt the following stepwise scheme.

Step 1 *(Variable Definition and Quadrification of Objective and Constraint Functions)*. For each $a \in A$ defined in (3), associate a variable $R[a]$ as above, and restrict

$$\underset{j=1}{\overset{n}{\pi}} \ell_j^{a_j} \leqslant R[a] \leqslant \underset{j=1}{\overset{n}{\pi}} u_j^{a_j} \quad \forall \text{ such variables } R[a]. \tag{4}$$

Replace each polynomial term $\phi_r(x)$ in PP by some quadratic expression of the form

$$\phi_r(x) \leftarrow \sum_{t \in T_r} \alpha_{rt} R[a^{t_1}]R[a^{t_2}], \text{ where } \{a^{t_1}, a^{t_2}\} \subseteq A, \text{ and where}$$
$$x^{(a^{t_1}+a^{t_2})} \equiv \underset{j \in J_{rt}}{\pi} x_j \forall t \in T_r, \quad r = 0,1,\ldots,R. \tag{5}$$

Step 2 (*Quadrification Constraints*). To establish the required inter-relationships among the $R[\cdot]$ variables, include all possible quadratic identities of the following form, where $R[0,\ldots,0] \equiv 1$.

$$R[a^1]R[a^2] = R[a^3]R[a^4] \quad \forall \{a^1,a^2,a^3,a^4\} \subseteq A \text{ such that}$$
$$a^1 + a^2 = a^3 + a^4 \leqslant S \equiv (S_1,\ldots,S_n), \text{ with}$$
$$\sum_{j=1}^{n} (a_j^1 + a_j^2) = \sum_{j=1}^{n} (a_j^3 + a_j^4) \leqslant \delta. \tag{6}$$

Step 3 (*Reformulation Phase of RLT*). Construct all possible pairwise products (including self-products) of the bound-factors defined in (4), so long as for any quadratic term $R[a^1]R[a^2]$ thus produced, we preserve the maximum degree δ, i.e., $\sum_{j=1}^{n}(a_j^1 + a_j^2) \leqslant \delta$.

Step 4 (*Linearization Phase of RLT*). Substitute a variable for each distinct product of the type $R[a^1]R[a^2]$. However, noting the relationship (6), we can substitute in this phase a variable $W[a^1 + a^2]$ for each distinct product of the type $R[a^1]R[a^2]$, so that if $a^1 + a^2 = a^3 + a^4$, the same linearizing variable would be substituted in place of $R[a^1]R[a^2]$ as would be for $R[a^3]R[a^4]$. This would hence render (6) redundant.

The net effect of this process is that at Step 1, each distinct polynomial term of the form $\pi_{j\in J}\, x_j$, for some $J \subseteq \bar{N}$, in the objective function and the constraints of Problem PP is replaced by a single variable $W[a]$, where a_j is the number of times the index j appears in J. Note that by (2), this variable $W[a]$ is precisely the variable X_J used in LP(PP), and so, the linearization phase applied to the quadrified objective function and constraints of PP would result in producing precisely (1a) and (1b), respectively. In addition, using the same linearization substitution as in (2), the bound-factor product constraints of Step 3, along with constraints (4), would produce the following restrictions.

$$\left[\left(\mathop{\pi}_{j=1}^{n} x_j^{a_j^1} - \mathop{\pi}_{j=1}^{n} \ell_j^{a_j^1}\right)\left(\mathop{\pi}_{j=1}^{n} x_j^{a_j^2} - \mathop{\pi}_{j=1}^{n} \ell_j^{a_j^2}\right)\right]_\ell \geqslant 0$$

and

$$\left[\left(\mathop{\pi}_{j=1}^{n} u_j^{a_j^1} - \mathop{\pi}_{j=1}^{n} x_j^{a_j^1}\right)\left(\mathop{\pi}_{j=1}^{n} u_j^{a_j^2} - \mathop{\pi}_{j=1}^{n} x_j^{a_j^2}\right)\right]_\ell \geqslant 0$$

for all distinct *unordered* pairs $\{a^1, a^2\} \subseteq A$ such that

$$\sum_{j=1}^{n} (a_j^1 + a_j^2) \leqslant \delta, \tag{7a}$$

$$\left[\left(\mathop{\pi}_{j=1}^{n} x_j^{a_j^1} - \mathop{\pi}_{j=1}^{n} \ell_j^{a_j^1}\right)\left(\mathop{\pi}_{j=1}^{n} u_j^{a_j^2} - \mathop{\pi}_{j=1}^{n} x_j^{a_j^2}\right)\right]_\ell \geqslant 0$$

for all distinct *ordered* pairs $\{a^1, a^2\} \subseteq A$ such that $\displaystyle\sum_{j=1}^{n} (a_j^1 + a_j^2) \leqslant \delta,$ (7b)

and

$$\mathop{\pi}_{j=1}^{n} \ell_j^{a_j} \leqslant \left[\mathop{\pi}_{j=1}^{n} x_j^{a_j}\right]_\ell \leqslant \mathop{\pi}_{j=1}^{n} u_j^{a_j} \quad \forall a \in A. \tag{7c}$$

Hence, the *all-encompassing* linear programming RLT relaxation that would result from simultaneously considering all possible quadrification transformations and then applying RLT to this as above, is given as follows.

$$\textbf{LP}(\overline{\textbf{QPP}}) : \ \text{Minimize } \{[\phi_0(x)]_\ell : \ \text{Constraints (1b) and (7)}\}. \tag{8}$$

REMARK 1. Note that we could have considered all possible bound factor products composed from (4) at Step 3, without the restriction that $\sum_{j=1}^{n} (a_j^1 + a_j^2) \leqslant \delta$ at the possible expense of increasing the degree of the polynomial terms in (7) to $\delta' > \delta$. If this is done, then the same dominance result below holds true by correspondingly also increasing δ to δ' in (1c) of LP(PP). ■

3. Dominance of LP(PP) over LP($\overline{QPP}$)

In the following discussion, we show that $\nu[LP(PP)] \geqslant \nu[LP(\overline{QPP})]$. In order to prove this dominance result, we first show that the defined terms of the type $(\pi_{j=1}^{n} x_j^{a_j} - \pi_{j=1}^{n} \ell_j^{a_j})$ and $(\pi_{j=1}^{n} u_j^{a_j} - \pi_{j=1}^{n} x_j^{a_j})$ as well as their products, can each be expressed as a sum of nonnegative multiples of ordinary bound-factor and nonnegative variable products. Hence, a linearization of products of such compound factors can likewise be expressed as a sum of linearizations of the latter type of ordinary nonnegative bound-factor and variable products. Then, by showing that such latter products are themselves implied by the RLT constraints defining LP(PP), we will establish the dominance results.

PROPOSITION 1. *The terms* $\pi_{j=1}^{n}(x_j^{p_j} - \ell_j^{p_j})$ *and* $\pi_{j=1}^{n}(u_j^{p_j} - x_j^{p_j})$*, where* $p \in Z_+^n$, $\sum_{j=1}^{n} p_j \leqslant \delta$*, as well as the term* $\pi_{j=1}^{n}(x_j^{p_j} - \ell_j^{p_j})\pi_{j=1}^{n}(u_j^{q_j} - x_j^{q_j})$*, where* $p, q \in Z_+^n$, $\sum_{j=1}^{n}(p_j + q_j) \leqslant \delta$*, can each be written as a sum of nonnegative multiples of terms of the type*

$$\pi_{j \in J_1}(u_j - x_j) \pi_{j \in J_2}(x_j - \ell_j) \pi_{j \in J_3} x_j \text{ where } (J_1 \cup J_2 \cup J_3) \subseteq \bar{N}, \quad |J_1 \cup J_2 \cup J_3| \leqslant \delta. \tag{9}$$

Proof. By the binomial expansion, we know that,

$$(y+a)^r = \binom{r}{0} y^r a^0 + \binom{r}{1} y^{r-1} a^1 + \cdots + \binom{r}{r-1} y^1 a^{r-1} + \binom{r}{r} y^0 a^r. \tag{10}$$

Hence, putting $a = \ell_j, y = (x_j - \ell_j)$, and $r = p_j$, for $j \in N$, in (10), we get

$$(x_j^{p_j} - \ell_j^{p_j}) = (x_j - \ell_j)^{p_j} + p_j \ell_j (x_j - \ell_j)^{p_j - 1} + \frac{p_j(p_j-1)}{2} \ell_j^2 (x_j - \ell_j)^{p_j-2} + \cdots + p_j \ell_j^{p_j-1}(x_j - \ell_j). \tag{11}$$

Similarly, putting $a = x_j, y = (u_j - x_j), r = p_j$, for $j \in N$ in (10), we get

$$(u_j^{p_j} - x_j^{p_j}) = (u_j - x_j)^{p_j} + p_j x_j (u_j - x_j)^{p_j - 1} + \frac{p_j(p_j-1)}{2} x_j^2 (u_j - x_j)^{p_j-2} + \cdots + p_j x_j^{p_j-1}(u_j - x_j). \tag{12}$$

The assertion of the lemma is now evident from (11) and (12), and this completes the proof. ■

PROPOSITION 2. *For any* $p \in Z_+^n$ *such that* $\sum_{j=1}^n p_j \leqslant \delta$, *the terms of the type* $(\pi_{j=1}^n x_j^{p_j} - \pi_{j=1}^n \ell_j^{p_j})$ *and* $(\pi_{j=1}^n u_j^{p_j} - \pi_{j=1}^n x_j^{p_j})$ *can each be expressed as a sum of nonnegative multiples of terms of the following form:*

$$\underset{j \in J_1}{\pi}(u_j - x_j) \underset{j \in J_2}{\pi}(x_j - \ell_j) \underset{j \in J_3}{\pi} x_j$$
$$\textit{where } (J_1 \cup J_2 \cup J_3) \subseteq \bar{N},\ |J_1 \cup J_2 \cup J_3| \leqslant \delta. \tag{13}$$

Proof. First, note that $(\pi_{j=1}^n x_j^{p_j} - \pi_{j=1}^n \ell_j^{p_j})$ can be expressed in terms of nonnegative multiples of products of factors $(x_j^{p_j} - \ell_j^{p_j}), j = 1, \ldots, n$, by inductively applying

$$\left(\underset{j=1}{\overset{n}{\pi}} x_j^{p_j} - \underset{j=1}{\overset{n}{\pi}} \ell_j^{p_j}\right) = \left(\underset{j=1}{\overset{n-1}{\pi}} x_j^{p_j} - \underset{j=1}{\overset{n-1}{\pi}} \ell_j^{p_j}\right)(x_n^{p_n} - \ell_n^{p_n}) + (x_n^{p_n} - \ell_n^{p_n}) \underset{j=1}{\overset{n-1}{\pi}} \ell_j^{p_j} + \ell_n^{p_n}\left(\underset{j=1}{\overset{n-1}{\pi}} x_j^{p_j} - \underset{j=1}{\overset{n-1}{\pi}} \ell_j^{p_j}\right).$$

In a similar fashion, $\left(\pi_{j=1}^n u_j^{p_j} - \pi_{j=1}^n x_j^{p_j}\right)$ can be expressed in terms of nonnegative multiples of products of factors $(u_j^{p_j} - x_j^{p_j})$ and $x_j^{p_j}$, $j = 1, \ldots, n$, by inductively applying

$$\left(\underset{j=1}{\overset{n}{\pi}} u_j^{p_j} - \underset{j=1}{\overset{n}{\pi}} x_j^{p_j}\right) = u_n^{p_n}\left(\underset{j=1}{\overset{n-1}{\pi}} u_j^{p_j} - \underset{j=1}{\overset{n-1}{\pi}} x_j^{p_j}\right) + (u_n^{p_n} - x_n^{p_n}) \underset{j=1}{\overset{n-1}{\pi}} x_j^{p_j}.$$

Hence, having done this, the assertion of Proposition 2 now follows by Proposition 1. This completes the proof. ∎

To prove the dominance result, we need one additional intermediary step, relating the constraints of type $[(13)]_\ell \geqslant 0$ with the regular RLT constraints (1c). Let us first define the following family of sets of constraints to identify all possible constructs of type (13).

$$\Omega_{s,\delta'} \equiv \left\{ \left[\underset{j \in J_1}{\pi}(u_j - x_j) \underset{j \in J_2}{\pi}(x_j - \ell_j) \underset{j \in J_3}{\pi} x_j\right]_\ell \geqslant 0 \right.$$
$$\forall \text{ distinct ordered triplets } (J_1, J_2, J_3) \quad \text{where}$$
$$(J_1 \cup J_2 \cup J_3) \subseteq \bar{N},\ |J_1 \cup J_2 \cup J_3| = \delta',\ |J_3| = s \tag{14}$$

for each $0 \leqslant s \equiv |J_3| \leqslant \delta' \leqslant \delta$. Notice that $\Omega_{0,\delta}$ is the set of regular RLT constraints (1c). Let us denote the feasible region defined by the constraint set $\Omega_{s,\delta'}$ by $\bar{\Omega}_{s,\delta'}$.

PROPOSITION 3.
(a) For any $0 \leqslant s \leqslant \delta$, $\bar{\Omega}_{s,\delta'} \supseteq \bar{\Omega}_{s,\delta'+1}$ *for all* $\delta' \in \{s, \ldots, \delta - 1\}$.
(b) For any $\delta' \in \{s, s+1, \ldots, \delta\}$, $\bar{\Omega}_{s-1,\delta'} \subseteq \bar{\Omega}_{s,\delta'}$, *for all* $1 \leqslant s \leqslant \delta$.

Proof. For any $0 \leqslant s = |J_3| < \delta$ and $\delta' \in \{s, \ldots, \delta - 1\}$, consider the following constraint from $\Omega_{s,\delta'}$:

$$\left[\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0. \tag{15}$$

Surrogating the following two constraints from the set $\Omega_{s,\delta'+1}$,

$$\left[(x_t - \ell_t)\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0$$

and

$$\left[(u_t - x_t)\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0 \tag{16}$$

where $t \in \{1, \ldots, n\}$, we obtain, that the sum equals $(u_t - \ell_t)$ times (15), hence implying (15). Therefore, we have $\bar{\Omega}_{s,\delta'} \supseteq \bar{\Omega}_{s,\delta'+1}$. This proves part (a).

To prove part (b), for any $1 \leqslant s \leqslant \delta' \leqslant \delta$ and $t \in \{1, \ldots, n\}$, consider the following constraint from $\Omega_{s,\delta'}$:

$$\left[x_t\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0 \tag{17}$$

for any $|J_3| = s - 1$, $J_1 \cup J_2 \cup J_3 \cup \{t\} \subseteq \bar{N}$, $|J_1 \cup J_2 \cup J_3 \cup \{t\}| = \delta'$. Notice that (17) is well defined, since $s = |J_3 \cup \{t\}| \geqslant 1$. We can obtain (17) by a particular surrogate of the following constraint from $\Omega_{s-1,\delta'}$,

$$\left[(x_t - \ell_t)\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0 \tag{18a}$$

with the following constraint from $\Omega_{s-1,\delta'-1}$, where the latter constraint has been shown to be implied by the constraints in $\Omega_{s-1,\delta'}$, in part (a) of this proposition,

$$\left[\underset{j\in J_1}{\pi}(u_j - x_j)\underset{j\in J_2}{\pi}(x_j - \ell_j)\underset{j\in J_3}{\pi} x_j\right]_\ell \geqslant 0. \tag{18b}$$

Using $\ell_t \geqslant 0$ as the weight for (18b), we have that (18a) + ℓ_t(18b) = (17). Hence, each constraint in $\Omega_{s,\delta'}$, is implied by those in $\Omega_{s-1,\delta'}$, and this completes the proof. ■

THEOREM 1. $\nu[LP(PP)] \geqslant \nu[LP(\overline{QPP})]$.

Proof. By Proposition 2, the compound factors $(\pi_{j=1}^n x_j^{p_j} - \pi_{j=1}^n \ell_j^{p_j})$ and $(\pi_{j=1}^n u_j^{p_j} - \pi_{j=1}^n x_j^{p_j})$ can each be expressed as a sum of nonnegative multiples of the terms of the form (13). Hence, so can the pairwise products of these compound factors. Consequently, constraints (7) can be obtained as surrogates of the constraints from the combined set $\{\Omega_{s,\delta'}, 0 \leqslant s \leqslant \delta' \leqslant \delta\}$. Since $\bar{\Omega}_{0,\delta} \subseteq \bar{\Omega}_{s,\delta'}$ for all $0 \leqslant s \leqslant \delta' \leqslant \delta$ by Proposition 3, the RLT constraints (7) of Problem $LP(\overline{QPP})$ are all implied by those in $\Omega_{0,\delta}$, where the latter are the RLT constraints (1c) of Problem

Table I. Comparison of $\nu[LP(PP)]$ versus $\nu[LP(\overline{QPP})]$.

Problem	(n, δ)	Known optimal value	$\nu[LP(PP)]$	$\nu[LP(\overline{QPP})]$
Problem 1	(1, 3)	−4.5	−4.5	−9
Problem 2	(1, 4)	0	−875	−6375.0
Problem 3	(1, 6)	7.0	−17385.0	−2,292,825.0
Problem 4	(2, 4)	−5.50796	−6.750	−6.9867
Problem 5	(2, 4)	−118.705	−8108.0	−54764.0
Problem 6	(2, 4)	−16.7389	−28.5	−29.0

$LP(PP)$. Hence, from (1) and (8), we have $\nu[LP(PP)] \geqslant \nu[LP(\overline{QPP})]$. This completes the proof. ∎

Notice that in establishing the above result, we have not assumed that the restrictions a_j^1, $a_j^2 \leqslant s_j$, $j = 1, \dots, n$, are imposed when generating the bound-factor products in (7). The removal of this restriction from (7) results in the generation of additional RLT constraints to be included in $LP(\overline{QPP})$ upon linearization. Therefore, the above dominance result holds even when considering this potentially tighter linear programming relaxation than the all-encompassing quadrified relaxation $LP(\overline{QPP})$.

4. A Computational Comparison of the Alternative Relaxations

To numerically illustrate the dominance of Theorem 1, we compare $\nu[LP(PP)]$ and $\nu[LP(\overline{QPP})]$ empirically below, using some test problems from the literature. Problems 1–4 are from Visweswaran and Floudas [13], and Problems 5 and 6 are from Ryoo and Sahinidis [10]. All the problems are scaled so that $(\ell, u) = (0, 1)$, where ℓ, u are the vectors of lower and upper bounds, respectively, on the variables. Table I presents the results obtained. In particular, the dominance is quite significant for Problems 2, 3 and 5. Also, in light of Remark 1, we have used $\delta' = 4$ in generating the RLT relaxations for Problem 1. However, even when we use only the RLT constraints of order $\delta = 3$ for $LP(PP)$, $LP(PP)$ yields a lower bound of -6, which still dominates the lower bound obtained via $LP(\overline{QPP})$. (This also illustrates that the bound via $LP(PP)$ can be enhanced by generating RLT constraints of order greater than δ.)

To examine the consequence of strengthening the relaxation $LP(\overline{QPP})$ even further beyond the intersection of all possible quadrified representations as mentioned after the proof of Theorem 1, we removed the restrictions $a_j^1, a_j^2 \leqslant s_j$, $j = 1, \dots, n$, from (7) for Problems 2 and 5. Although the inclusion of the consequent additional constraints improved the respective lower bounds to –5241.67 and –46654 for these problems, the difference from $\nu[LP(PP)]$ still remains significant.

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References

1. Al-Khayyal, F.A., C. Larson, and T. Van Voorhis (1994), A Relaxation Method for Nonconvex Quadratically Constrained Quadratic Programs, Working Paper, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332.
2. Floudas, C.A. and V. Visweswaran (1993), A Primal-Relaxed Dual Global Optimization Approach: Theory, *JOTA* **78**, 187–225.
3. Floudas, C.A. and V. Visweswaran (1990), Global Optimization Algorithm (GOP) for Certain Classes of Nonconvex NLP's I. Theory, *Computers and Chemical Engineering* **14**, 1397–1417.
4. Hansen, P. and B. Jaumard (1992), Reduction of Indefinite Quadratic Programs to Bilinear Programs, *Journal of Global Optimization* **2**(1), 41–60.
5. Hansen, P., B. Jaumard, and S. Lu (1991), An Analytical Approach to Global Optimization, *Mathematical Programming* **52**, 227–254.
6. Horst, R. (1986), A General Class of Branch-and-Bound Methods in Global Optimization with Some New Approaches for Concave Minimization, *JOTA* **51**, 277–291.
7. Horst, R. and H. Tuy (1993), *Global Optimization: Deterministic Approaches*, 2nd Ed., Springer-Verlag, Berlin, Germany.
8. McCormick, G.P. (1976), Computability of Global Solutions to Factorable Nonconvex Programs: Part I – Convex Underestimating Problems, *Mathematical Programming* **10**, 147–175.
9. Quesada, I. and I.E. Grossmann (1995), A Global Optimization Algorithm for Linear Fractional and Bilinear Programs, *Journal of Global Optimization* **6**, 39–76.
10. Ryoo, H.S. and N.V. Sahinidis (1995), Global Optimization of Nonconvex NLPs and MINLPs with Applications in Process Design, *Computers and Chemical Engineering* **19**(5), 551–566.
11. Sherali, H.D. and C.H. Tuncbilek (1992), A Global Optimization Algorithm for Polynomial Programming Problems using a Reformulation-Linearization Technique, *Journal of Global Optimization* **2**, 101–112.
12. Shor, N.Z. (1990), Dual Quadratic Estimates in Polynomial and Boolean Programming, *Annals of Operations Research* **25**, 163–168.
13. Visweswaran, V. and C.A. Floudas (1992), Unconstrained and Constrained Global Optimization of Polynomial Functions in One Variable, *Journal of Global Optimization* **2**, 73–99.